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# ON THE DEFECT INDICES OF LINEAR OPERATORS IN BANACH SPACE AND ON SOME GEOMETRIC QUESTIONS

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by

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Summary

The authors show that the theory of defect indices of Hermitian operators in a Hilbert space can be extended to the case of linear operators in a Banach space. For this purpose they use the idea of aperture (gap) of two subspaces with whose help it is convenient to establish in many cases the equality of the dimensions of the subspaces of the Banach space. The authors also arrive at several geometric conjectures.



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# ON THE DEFECT INDICES OF LINEAR OPERATORS IN BANACH SPACE AND ON SOME GEOMETRIC QUESTIONS*

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It has been shown not long ago [1,2] that the Carleman-Neumann theory of defect indices of Hermitian operators in a Hilbert space in its basic state can be extended to the case of arbitrary linear operators in that space.

In this paper we shall show that this theory permits the extension to the case of linear operators in a Banach space.

Just as in [2], we shall use for this purpose the idea of aperture of two subspaces with whose help it is convenient to establish in many cases the equality of the dimensions of the subspaces of the Banach space.

In connection with this concept the authors came to several geometric conjectures, which possibly present an interest in themselves.

Detailed examination of this concept, in the case of a Hilbert space led to the study of a spectral reflection operator generated by two subspaces, and various characteristic slopes of one subspace with respect to the other.

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## I. On the dimension number of a Banach Space

1. Let M be a subset of a Banach space E (real or complex) and suppose that the linear span M is dense in E. We shall call such subsets $M \subset E$ generators of E. Let $a_M$ be the cardinal number of the subset M.

The smallest cardinal number $a_M$, where M is the set in the aggregate generating E, is called the dimension number of E and is denoted by dim E.

If the dimension number of E is finite, then it is easy to see that it coincides with the maximum number of linearly independent elements belonging to E.

If dim E is infinite, then it coincides with the minimum of all the cardinal numbers of sets which are dense in E. In order to show this, it suffices to notice that the set of linear combinations of elements with rational coefficients contained in the set which generates E is dense in E.

We shall call a subset A of a set $B \subset E$ an $\alpha$-lattice B, if for arbitrary $x, y \in A$ we have $\|x-y\| \geq \alpha$. We say that the $\alpha$-lattice is maximal, if the $\alpha$-lattice A of the set B does not preserve the inequality for any other $\alpha$-lattice of B.

Thus, if A is a maximal $\alpha$-lattice of the set B, then for every $z \in B$

$$\rho(z, A) = \inf_{x \in A} \|z-x\| < \alpha .$$

Any $\alpha$-lattice can be extended to be maximal by the transfinite process; from this, in particular, follows the existence of a maximal lattice.

Lemma 1. Let E be an infinite dimensional Banach space. Then the cardinal number of every maximal $\alpha$-lattice A of the unit hyperball K with $0 < \alpha < 1$ coincides with dim E.

Proof: Let A be some maximal $\alpha$-lattice $(0 < \alpha < 1)$ of the unit hyperball K.

Assume that its cardinal number $a_A$ is less than dim E. Then the set L of linear combinations of elements of A with rational coefficients is not dense in E. It follows by a well-known lemma of Riesz that for arbitrary $\epsilon > 0$ we can find an element $x \in E$ $(\| x \| = 1)$ such that, $\rho(x,L) \geq 1 - \epsilon$. Choosing $\epsilon < 1 - \alpha$ leads to a contradiction, since A is a maximal $\alpha$-lattice. Thus $a_A$ is not smaller than dim E.

On the other hand, $a_A$ cannot be larger than dim E, since in the neighborhood of radius $\alpha/2$ the elements of the $\alpha$-lattice don't intersect, but in each of them there is at least one element of every set which is dense in E.

This concludes the proof of the lemma.

Remark: We can easily convince ourselves that Lemma 1 holds when the hyperball K is replaced by the hypersphere S $(\| x \| = 1)$.

2. The following assertion is easily shown using Lemma 1.

Theorem 1. The dimension of the Banach space E is not larger than the dimension of the conjugate space $E^*$.

In particular, if the space E can be reflected, then

$$\dim E = \dim E^* .$$

Proof: If dim E is finite, then the assertion of the theorem is obvious.

Suppose dim E is infinite. Choose a transfinite maximal sequence of elements $\{x_\alpha\}$ ($\|x_\alpha\| = 1$) in E such that

$$\rho(x_\beta, L_\beta) > 1/2 ,$$

where $L_\beta$ is the linear span of elements $x_\alpha$ for which $\alpha < \beta$. The closure of the linear span of elements of the sequence is E, and the cardinal number of the set of indexes $\alpha$ of the sequence is dim E.

We now construct a sequence of functionals $\{f_\alpha\}$ ($\|f_\alpha\| = 1$) such that

$$|f_\alpha(x_\alpha)| > 1/2, \; f_\alpha(x) = 0 \text{ for } x \in L_\alpha .$$

Suppose $f_{\alpha'}$ and $f_{\alpha''}$ ($\alpha' < \alpha''$) are elements of the constructed sequence of functionals. Then

$$\|f_{\alpha'} - f_{\alpha''}\| \geq \frac{|(f_{\alpha'} - f_{\alpha''})(x_{\alpha'})|}{\|x_{\alpha'}\|} = |f_{\alpha'}(x_{\alpha'})| > 1/2 .$$

Consequently, the constructed transfinite sequence of functionals $\{f_\alpha\}$ is a part of some maximal 1/2-lattice of a unit hyperball of the space E* .

This concludes the proof of the theorem.

## II. The aperture of two subspaces

1. The aperture of two linear manifolds $L_1$ and $L_2$ of a Banach space E, designated by $\theta(L_1, L_2)$ (sec [2]), is defined satisfying the following equation

$$\theta(L_1, L_2) = \max \{\sup_{\substack{x \in L_1 \\ \|x\|=1}} \rho(x, L_2), \sup_{\substack{y \in L_2 \\ \|y\|=1}} \rho(y, L_1)\} . \tag{1}$$

Several evident properties of the aperture can be noted. First, we always have

$$0 \leq \theta(L_1, L_2) \leq 1 ,$$

and second,

$$\theta(\bar{L}_1, \bar{L}_2) = \theta(L_1, L_2) ,$$

where $\bar{L}_1$, $\bar{L}_2$ are the closures of $L_1$ and $L_2$ respectively.

Theorem 2. Let $L_1$ and $L_2$ be linear sets of a Banach space E and suppose

$$\theta(L_1, L_2) = a < 1 .$$

If one of the numbers dim $L_1$, dim $L_2$ *) is finite, then

$$\dim L_1 = \dim L_2 .$$

Proof: The theorem is equivalent to showing that if dim $L_1 = n$, dim $L_2 > n$, then $\theta(L_1, L_2) = 1$. In order to show the latter it suffices to show that there exists a unit vector $y(\|y\| = 1)$, in $L_2$ which is orthogonal to $L_1$, i.e., such that $\rho(y, L_1) = 1$; in this case we can assume without loss of generality that dim $L_1 = n + 1$. Let E denote the linear span of either $L_1$ or $L_2$. Assume first that the unit sphere $\|z\| = 1$ $(z \in E)$ is strictly convex, i.e., it doesn't contain segments. Then for each $z \in E$ there will exist only one projection in $L_1$, i.e., only one $x \in L_1$ through which the distance from z to $L_1 : \rho(z, L_1) = \|z - x\|$ will be attained. It's easily seen that the projection operator $x = \varphi(z)$ $(z \in E)$ is continuous and satisfies $\varphi(-z) = -\varphi(z)$. The orthogonality of z to $L_1$ shows that $\rho(z, L_1) = \|z - \varphi(z)\| = \|z\|$, i.e., $\varphi(z) = 0$ (by virtue of the uniqueness of projection).

*) By dim L it's understood dim $\bar{L}$.

If we now assume that on the unit sphere $S_2(\|y\| = 1)$ of the space $L_2$, the operator $\varphi(y)$ is different from zero, then by virtue of the compactness of $S_2$, it will be possible to assert that the operator

$$\psi(y) = \frac{\varphi(y)}{\|\varphi(y)\|}$$

is also continuous on $S_2$. But this operator continuously maps the n-dimensional sphere $S_2$ into the (n - 1)-dimensional sphere $S_1(\|x\| = 1)$ of the space $L_1$, and since in addition centrally symmetric points map into centrally symmetric points $(\varphi(-y) = -\psi(y))$, this is impossible )*.

Thus, the theorem is shown assuming that the sphere is strictly convex in E.

We shall reduce the general case to the one considered, by showing that for an arbitrary $\varepsilon > 0$ one can always construct a new norm $\|z\|_o$ in E such that

$$\|z\| \leq \|z\|_o \leq (1 + \varepsilon)\|z\| \quad (z \varepsilon E) \tag{2}$$

and such that the new sphere $\|z\|_o = 1$ which is strictly convex, i.e., such that for any two vectors $z_1, z_2 \varepsilon E$ with different directions

$$\|z_1 + z_2\|_o < \|z_1\|_o + \|z_2\|_o . \tag{3}$$

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)* We clarify, that the n-dimensional sphere in the Banach space E can be homeomorphically mapped into an n-dimensional (if E is real) or 2 n-dimensional (if E is a complex space) sphere of ordinary Euclidean space preserving the central symmetry of images of symmetric points. On the other hand, using a theorem of L. A. Lusternik and L. G. Shpirelman on categories of projection spaces (see [7]), it can easily be shown that if a k-dimensional Euclidean sphere is continuously mapped preserving the central symmetry into a m-dimensional Euclidean sphere, then $m \geq k$.

In fact, inequality (2) evidently implies the following inequality

$$\theta_o(L_1,L_2) \le (1+\varepsilon)\,\theta(L_1,L_2) ,$$

where $\theta_o(L_1,L_2)$ is the aperture between $L_1$ and $L_2$ corresponding to the norm $\|z\|_o$. Consider $L_1$ and $L_2$, then by showing $\theta_o(L_1,L_2) = 1$ we'll have by virtue of condition (3) that $\theta(L_1,L_2) = 1$ since $\varepsilon > 0$ is arbitrary.

We shall show now how one can proceed constructing the norm $\|z\|_o$.

Let $\|z\|_1$ be a norm in E, and corresponding to it the strictly convex sphere $\|z\|_1 = 1$; for example this norm can be determined by setting

$$\|\xi_1 e_1 + \xi_2 e_2 + \dots + \xi_m e_m\|_1 = \sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_m^2} ,$$

where $\{e_1, e_2, \dots, e_m\}$ is some basis in E.

Let $K(>0)$ be the maximum $\|z\|_1$ on the unit sphere $\|z\| = 1$, then we can write

$$\|z\|_1 \le K\|z\| \quad (z \in E) .$$

Since $\|z\|_1$ satisfies condition (3), it follows that for arbitrary $\delta > 0$ the following norm will also satisfy this condition

$$\|z\|_o = \|z\| + \delta\,\|z\|_1 .$$

For this norm inequality (2) will hold with $\varepsilon = \delta K$.

Consequently since $\delta > 0$ is arbitrary the required construction is achieved and the theorem is shown.

Theorem 3. Let $L_1$ and $L_2$ be linear manifolds of a Banach space E and let

$$\theta(L_1, L_2) < 1/2 .$$

Then

$$\dim L_1 = \dim L_2 .$$

Proof: By virtue of theorem 2, it follows that we need only consider the case when $\dim L_1$ and $\dim L_2$ are infinite.

Let $\theta(L_1, L_2) = 1/2 - b$ $(b > 0)$. In the unit hyperball K of the linear manifold $L_1$, construct a maximal $\alpha$-lattice A, where for instance $\alpha = 1 - b/2$. By this condition we have that for each element $x \in A \subset L_1$ there exists an element $y_x \in L_2$ such that $\|x - y_x\| \leq 1/2 - b/2$. If $x_1, x_2 \in A$ $(\|x_1 - x_2\| > 1 - b/2)$, then

$$\|y_{x_1} - y_{x_2}\| \geq \|x_1 - x_2\| - \|x_1 - y_{x_1}\| - \|x_2 - y_x\| > b/2 .$$

From this inequality it follows that the elements $y_x$, which correspond to elements x of the maximal $(1 - b/2)$-lattice A of the hyperball K, are contained in some $b/2$-lattice of a hyperball of radius 3/2 $(\|y_x\| \leq \|y_x - x\| + \|x\| < 3/2)$ of the linear set $L_2$.

It follows by virtue of lemma 1 that

$$\dim L_1 \leq \dim L_2 .$$

Interchanging the roles of $L_1$ and $L_2$, we come to the conclusion of the theorem.

2. Let L be a linear subset of E. The subspace $L^*\subset E^*$, consisting of all functionals which vanish for all elements of L, is called the orthogonal complement of L in $E^*$.

<u>Theorem 4</u>. Let $L_1$ and $L_2$ be two linear subsets of E, and let $L_1^*$, $L_2^*$ be their orthogonal complements in $E^*$.

Then the following holds:

$$\theta(L_1,L_2) = \theta(L_1^*,L_2^*).$$

<u>Proof</u>: By formula (1) we can write:

$$\theta(L_1,L_2) = \sup\{|g(x)|,\ |f(y)|\},$$

where the supremum is taken over all

$$x\in L_1,\ y\in L_2,\ f\in L_1^*,\ g\in L_2^*;\ \|x\| = \|y\| = \|f\| = \|g\| = 1.$$

Recall, that if the subspace $L^*\subset E^*$ is regularly closed (i.e., for every functional $f_o\overline{\in} L^*$, $f_o\in E^*$ there exists an element $x_o\in E$ such that $f_o(x_o) \neq 0$, but $f(x_o) = 0$ for all $f\in L^*$), then by a known theorem of Banach (see [3]) we have that for an arbitrary $f_o\overline{\in} L^*$ and $\varepsilon > 0$ there exists an element $x_o\in L$ ($\|x_o\| = 1$) such that

$$|f_o(x_o)| \geq \rho(f_o,L^*) - \varepsilon,\ f(x_o) = 0 \text{ for all } f\in L^*.$$

In this connection it's obvious that $|f_o(x_o)| \leq \rho(f_o,L^*)$ always holds.

Since the orthogonal complement $L^*$ of the linear set $L$ is, obviously, regularly closed, it follows on the basis of this theorem that

$$\rho(f, L^*) = \sup_{x \in L, \|x\|=1} |f(x)| .$$

This means that

$$\theta(L_1^*, L_2^*) = \sup \{|g(x)|, |f(y)|\} ,$$

where the supremum is taken over all

$$x \in L_1, \; y \in L_2, \; f \in L_1^*, \; g \in L_2^*; \; \|x\| = \|y\| = \|f\| = \|g\| = 1 .$$

This concludes the proof of the theorem.

The following assertion follows directly from theorems 3 and 4.

Theorem 5. Let $L_1$ and $L_2$ be two linear subsets in a Banach space $E$, let $L_1^*$ and $L_2^*$ be their orthogonal complements in $E^*$ .

Then, if

$$\theta(L_1, L_2) < 1/2,$$

then

$$\dim L_1^* = \dim L_2^* .$$

3. Let $L$ be a subspace of a Banach space $E$. It is known that the subspace conjugate with the quotient-space $E/L$ is equivalent to the subspace $E^* \subset L^*$.

If the space E/L is reflexive, (which occurs if, for example, E is reflexive, see [4]), then by theorem 1 dim E/L = dim L* .

With $\theta(L_1,L_2) < 1/2$ $(L_1,L_2 \subset E)$ it follows by virtue of theorem 5 that

$$\dim E/L_1 = \dim E/L_2 .$$

It turns out that this fact also holds for nonreflexive spaces.

<u>Theorem 6</u>. Let $L_1$ and $L_2$ be subspaces of a Banach space E and let

$$\theta(L_1,L_2) < 1/2 .$$

Then

$$\dim E/L_1 = \dim E/L_2 . \tag{4}$$

In case one of $\dim E/L_1$, $\dim E/L_2$ is finite, equation (4) follows from

$$\theta(L_1,L_2) < 1.$$

Proof: The second assertion of the theorem follows from the fact that a finite dimensional space is reflexive, and from theorems 2 and 4.

Assume now that $\dim E/L_1$ and $\dim E/L_2$ are infinite.

Denote by x the image of the element $x \in E$ under the mapping into the quotient space $E/L_1$, denote the image in $E/L_2$ by $\tilde{x}$ .

Let us assume that $\dim E/L_1 > \dim E/L_2$.

In the unit sphere $S_1$ of the space $E/L_1$ construct a maximal $(1 - \beta)$-lattice $A_1$ (where $\beta$ is an arbitrary positive number), i.e., the totality of elements $x \in E$, whose cardinal equals $\dim E/L_1$ (see remark 1), for which

$$\|\hat{x} - \hat{y}\| \geq 1 - \beta, \ \|\hat{x}\| = \|\hat{y}\| = 1, \ \hat{x}, \ \hat{y} \in A_1 \subset E/L_1 \ .$$

From the definition of the quotient space it follows that in every co-set $\hat{x} \in A_1$ we can choose an element $x \in E$, such that if $\gamma$ is an arbitrary positive number then $\|x\| \leq 1 + \gamma$.

Then from $\|\hat{x} - \hat{y}\| > 1 - \beta$ also follows that $\|x - y\| > 1 - \beta$. By assuming $\dim E/L_1 > \dim E/L_2$, and also by lemma 1 we have that the elements $x$ cannot be contained in any $\alpha$-lattice of a hypersphere of radius $1 + \gamma$ ($\|\tilde{x}\| < 1 + \gamma$) of the space $E/L_2$. This means that for an arbitrary positive $\alpha$ there exist $x, y \in E$ ($\|x\|$, $\|y\| < 1 + \gamma$; $\hat{x}$, $\hat{y} \in A_1$) such that $\|\hat{x} - \hat{y}\| < \alpha$. The last inequality means that there exists an element $z \in L_2$ such that $\|x - y - z\| < \alpha$, whence

$$1 - \beta - \alpha < \|z\| < 2 + 2\gamma + \alpha \ .$$

Then

$$\rho\left(\frac{z}{\|z\|}, L_1\right) \geq \frac{\rho(x-y, L_1) - \|x-y-z\|}{\|z\|} > \frac{1 - \beta - \alpha}{2 + 2\gamma + \alpha} \ .$$

By virtue of the arbitrariness of $\alpha, \beta, \gamma$ it follows from the inequality above that

$$\theta(L_1, L_2) \geq 1/2 \ .$$

This concludes the proof of the theorem.

## III. Several Questions of the Geometry of a Hilbert Space

1. It is not clear whether the conclusion of theorem 2 holds without assuming finiteness of dimension of one of the linear manifolds (i.e., can we replace the condition $\theta(L_1,L_2) < 1/2$ in theorem 3 by the condition $\theta(L_1,L_2) < 1$?).

In the case when E is a Hilbert space, the posed question, as already remarked (see [2]), is trivially solvable.

If $L_1,L_2$ are two subspaces of arbitrary dimension in a Hilbert space E (we shall only consider such in this paragraph) which satisfy the following condition $\theta(L_1,L_2) < 1$, then in any one of them there does not exist a non-zero vector which is orthogonal to the other space.

We shall say that any two subspaces $L_1,L_2$ of a Hilbert space E, which satisfy the last condition, form a proper pair.

One can easily be convinced of the validity of the following proposition.

The subspaces $L_1,L_2 \subset E$, which form a proper pair, have identical dimensions.

In fact, $P_{L_2} L_1$ [1]) is dense in $L_2$, since otherwise we can find a vector $h \neq 0$ in $L_2$ which is orthogonal to $P_{L_2} L_1) \leq \dim L_1$. Analogously, $\dim L_1 \leq \dim L_2$ and it therefore follows that

$$\dim L_1 = \dim L_2 .$$

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[1]) If L is a subspace of E, then $P_L$ denotes an operator which projects orthogonally on L.

2. The aperture in a Hilbert space is only a characteristic ordering relative to two subspaces. Below, we shall derive a more complete characteristic ordering relative to subspaces.

Let M be a subspace of E. An element $x^* \in E$ is said to be the spectral reflection of an element $x \in E$ with respect to M, if $x^* = 2P_M x - x$. If we introduce into consideration the orthogonal complement N of M

$$E = M \oplus N$$

then $x^*$ can also be denoted by

$$x^* = P_M x - P_N x \ .$$

Thus $x^* = Sx$, where $S = P_M - P_N$ (from which, in particular, $(x^*)^* = x$).

We see that the spectral reflection operator S with respect to M, is first unitary, and second possesses an inverse: $S^2 = I$ (i.e., $(x^*)^* = x$).

On the other hand, it's easy to see that any operator S which is unitary and possesses an inverse ($S^2 = I$) is a spectral operator - a spectral reflection operator with respect to some subspace M.

In fact for such an operator we have

$$S = P - Q, \ P + Q = I$$

where

$$P = \frac{I + S}{2} \quad , \quad Q = \frac{I - S}{2}$$

are orthogonal projection operators ($P^2 = P = P^*$, $Q^2 = Q = Q^*$).

Suppose $L_1, L_2 \subset E$ are two subspaces which are spectral reflections of each other with respect to M, i.e., $SL_1 = (P_M - P_N)L_1 = L_2$, then $(P_N - P_M)L_1 = L_2$, i.e., $L_1$ and $L_2$ are also spectral reflections of each other with respect to N.

Since, if $x \in L_1$, the projection $P_M x = \frac{1}{2}(x + Sx) \in L_1 + L_2$, we conclude that if $L_1, L_2$ are spectral reflections of each other with respect to M, then they are also spectral reflections of each other with respect to the intersection $M_1 = M \;\; \overline{L_1 + L_2}$ and vice versa.

In order that the subspaces $L_1$ and $L_2$ be spectral reflections of each other, it is evidently necessary that $\dim L_1 = \dim L_2$. However, this condition is not sufficient. In fact, suppose the infinite dimensional subspaces $L_1$ and $L_2$ form a proper pair.

By virtue of the following theorem, $L_1$ and $L_2$ will be spectral reflections with respect to some M.

Let e be a vector, orthogonal to $L_1$, and $L_2'$ the linear span of $L_2$ and e. Then as before $\dim L_1 = \dim L_2'$ ; moreover $L_1$ cannot be the spectral reflection of $L_2'$ with respect to any M', since by assuming the contrary, we would have vector $e^*$ in $L_2$ orthogonal to $L_2'$, and hence to $L_2$, where e is the spectral reflection of $e^*$ ($\|e^*\| = \|e\| > 0$).

As a consequence of all this the following assertion is of interest.

Theorem 7. If the subspaces $L_1, L_2 \subset E$ form a proper pair, then they are spectral reflections of each other with respect to some $M \subset \overline{L_1 + L_2}$ .

Proof. Consider the operator $P_{L_1} P_{L_2}$ in the invariant subspace $L_1$ .

This operator is self-adjoint, and furthermore, strictly positive:

$$(P_{L_1} P_{L_2} x,x) = (P_{L_2} x,x) = \|P_{L_2} x\|^2 > 0 \qquad (x \in L_1, x \neq 0),$$

since by the condition, there are no vectors ($\neq 0$) in $L_1$ which are orthogonal to $L_2$. Consequently, there exists a self-adjoint operator H (which is not uniquely determined) in $L_1$, such that

$$H^2 x = P_{L_1} P_{L_2} x \qquad (x \in L_1) .$$

This operator satisfies

$$\|Hx\|^2 = (H^2 x,x) = \|P_{L_2} x\|^2 > 0 \text{ for } x \neq 0 \qquad (x \in L_1) .$$

Therefore the self-adjoint operator $H^{-1}$ has a meaning in $L_1$; its domain of definition D (the set of all values of the operator H) is dense in $L_1$ .

Let us introduce into consideration the operator S which acts from D to $L_2$, by setting:

$$Sx = P_{L_2} H^{-1} x \qquad (x \in D) .$$

We have

$$(Sx,y) = (P_{L_1} P_{L_2} H^{-1} x,y) = (P_{L_1} P_{L_2} H^{-1} x,y) = (Hx,y) \qquad (x \in D,\ y \in L_1) .$$

Since $(Hx,y) = (x,Hy)$, then for any $x,y \in D$

$$(Sx,y) = (x,Sy) = (Hx,y) \ . \tag{5}$$

It follows that, for any $x,y \in D$:

$$(Sx,Sy) = (P_{L_2} H^{-1} x, Sy) = (H^{-1} x, Sy) = (HH^{-1} x, y) = (x,y) \ .$$

Thus we conclude that S isometrically maps D into $L_2$. Since $SD = P_{L_2} H^{-1} D = P_{L_2} L_1$, then SD is dense in $L_2$.

We extend the operator S to all of $L_1$ preserving the isometry, we use the same notation S for the extended operator. The extended operator maps isometrically all of $L_1$ onto a closed part of $L_2$, and hence onto all $L_2$, and equation (5) holds now for all $x,y \in L_1$.

Relation (5) is equivalent to the following

$$(Sx_1, S^{-1} x_2) = (x_1, x_2) \qquad (x_1 \in L_1, x_2 \in L_2)$$

and therefore, for any $x_1 \in L_1, x_2 \in L_2$:

$$\| Sx_1 + S^{-1} x_2 \| = \| x_1 + x_2 \| \ .$$

From this we conclude that the operator S admits an isometric extension $\widetilde{S}$ to all $E = \overline{L_1 + L_2}$ , which is defined on $L_1 + L_2$ by the equation

$$\widetilde{S}(x_1 + x_2) = Sx_1 + S^{-1} x_2 \ .$$

Since $\widetilde{S}x_1 = y_2 \varepsilon L_2$ and $\widetilde{S}x_2 = y_1 \varepsilon L_1$, then

$$\widetilde{S}^2(x_1 + x_2 = \widetilde{S}(y_1 + y_2) = x_1 + x_2 \; .$$

Thus $\widetilde{S}^2 = I$; since, in addition, $\widetilde{S}L_1 = L_2$, $\widetilde{S}L_2 = L_1$, then $\widetilde{S}E \supset L_1 + L_2$ and consequently $\widetilde{SE} \dotplus E$.

Thus $\widetilde{S}$ is a spectral reflection operator in E, which maps $L_1$ onto $L_2$. This proves the theorem.

Remark. We leave it to the reader to show that the indicated construction of the spectral operator $\widetilde{S}$ in $E = \overline{L_1 + L_2}$ exhausts all spectral operators reflecting one onto the other $L_1$ and $L_2$ .

There exists among these operators an operator $S_+$, which corresponds to a positive Hermitian operator $H_+$ and which characterizes the geometric conditions that for this operator the angle between any $x \varepsilon L_1$ $(x \neq 0)$ and its image $x^* = S_+ x \varepsilon L_2$ is not greater than $\pi/2$:

$$(x, x^*) = (x, S_+ x) = (x, H_+ x) > 0 \; .$$

Any other Hermitian operator H in our class can be given by $H = 0^{-1} H_+$, where 0 is a unitary operator in $L_1$. This follows since in our class the condition $H^2 = H_+^2$ , means that $\|Hx\| = \|H_+ x\| \quad (x \varepsilon L_1)$ .

Since $H^* = H$, then also

$$H = 0^{-1} H_+ = H_+ 0 \; . \tag{6}$$

Consequently,

$$H_+^2 = H^2 = 0^{-1} H_+^2 \, 0 \ ,$$

i.e., 0 commutes with $H_+^2$, and consequently, with $H_+$ .

But then by (6): $0^2 = I$.

Thus, the general form of the spectral reflection $x \to x^*$ of the spaces $L_1$ onto $L_2$ is given by the formula

$$x^* = \overline{P_{L_2} H_+^{-1} 0 \ x} \qquad (x \in L_1) \ ,$$

where $H_+$ is a positive self-adjoint operator acting in $L_1$ which when squared yields $P_{L_1} P_{L_2}$ and 0 is an arbitrary reflection operator in $L_1$ which commutes with $H_+$ .

2. Now let $L_1, L_2$ be arbitrary subspaces in E. As previously, we denote by $H_+$, which is given by the formula $H_+^2 x = P_{L_1} P_{L_2} x \quad (x \in L_1)$, a positive Hermitian operator in $L_1$, which we shall also denote by $H_{12}$. Since the spectrum of the operator $H_{12}$ lies in the interval (0,1):

$$0 \leq (H_{12}^2 x, x) = (P_{L_2} x, x) = \|P_{L_2} x\|^2 \leq \|x\|^2 \qquad (x \in L_1),$$

then, there exists one and only one self-adjoint operator $\Phi_{12}$ in $L_1$ with spectrum in the interval $(0, {}^{\pi}/2)$, such that

$$\cos \Phi_{12} = H_{12} \ .$$

It is natural to call the spectrum of the operator $\Phi_{12}$ as the spectrum of the angles of inclination of $L_1$ and $L_2$. Let $\varphi_{12}^{(M)}$ and $\varphi_{12}^{(m)}$ $(0 \leq \varphi_{12}^{(m)} \leq {}^{\pi}/2)$ correspond to the biggest and smallest points of this spectrum.

For any unit vector x ($\|x\| = 1$) the angle of inclination to $L_2$ $\varphi(0 \le \varphi \le \pi/2)$, is determined by the equation

$$\sin\varphi = \rho(x, L_2) = \sqrt{1 - \|P_{L_2} x\|^2} .$$

Obviously,

$$\sin^2\varphi = 1 - (H_+^2 x, x) = (x,x) - (\cos^2 \Phi_{12} x, x) = (\sin^2 \Phi_{12} x, x) .$$

Thus for the magnitude $\theta_{12} = \sup\limits_{\substack{x \in L_1 \\ x = 1}} \rho(x, L_2)$ we have the following interpretation:

$$\theta_{12} = \sup \sin\varphi = \sin\varphi_{12}^{(M)} .$$

Interchanging the roles of $L_1$ and $L_2$, we obtain, correspondingly, the operators $H_{21}$ and $\Phi_{21}$ and the magnitudes $\theta_{21}$, $\varphi_{21}^{(M)}$, $\varphi_{21}^{(m)}$ .

In the case if the subspaces $L_1$ and $L_2$ form a proper pair, it follows by theorem 7 that they are spectral reflections of each other and, consequently,

$$\theta_{12} = \theta_{21};\ \varphi_{12}^{(M)} = \varphi_{21}^{(M)} = \varphi^{(M)} . \qquad (7)$$

Thus the aperture of a proper pair of subspaces $L_1$ and $L_2$ can be interpreted, as the sine of the maximal angle $\varphi^{(M)}$ between the subspaces $L_1$ and $L_2$:

$$\theta(L_1, L_2) = \sin\varphi^{(M)} .$$

The equalities in (7) are consequences of a more general proposition.

The angular operators $\Phi_{12}$ and $\Phi_{21}$ are unitarily-equivalent when $L_1$ and $L_2$ are a proper pair of subspaces.

In fact, let S be a spectral reflection operator ($SL_1 = L_2$). Let $x^* = Sx$ ($x \in L_1$) and $y = P_{L_2} x$, then $y^* = P_{L_1} x^*$, since if $x - y$ is orthogonal to $L_2$, $x^* - y^*$ is orthogonal to $L_1$. Thus, if $z = P_{L_1} P_{L_2} x$, then $z^* = P_{L_2} P_{L_1} x^*$, i.e.,

$$SH_{12}^2 S^{-1} = H_{21}^2, \quad SH_{12} S^{-1} = H_{21}$$

and

$$S\Phi_{12} S^{-1} = \Phi_{21} .$$

We remark that in case $L_1$ and $L_2$ are a proper pair the spectral reflection operator S can be written in the following form:

$$Sx = \overline{P_{L_2} (\cos \Phi_{12})^{-1} x} \qquad (x \in L_1) .$$

In case the subspaces $L_1$ and $L_2$ do not form a proper pair, then they can be decomposed in a natural fashion into orthogonal sums

$$L_1 = L_1' \oplus L_{12} \; ; \quad L_2 = L_2' \oplus L_{21} ,$$

where $L_{12}$ - (correspondingly $L_{21}$) is a subspace of elements of $L_1 (L_2)$, orthogonal to $L_2 (L_1)$. Then it is easily seen that $L_1$ and $L_2'$ form a proper pair, and in order that there exists a spectral reflection of $L_1$ onto $L_2$ it is necessary and sufficient that $\dim L_{12} = \dim L_{21}$.

We remark that the subspace $L_{12}$ for the operator $H_{12}(\Phi_{12})$ is an eigen-subspace, corresponding to the eigenvalue $0(\pi/2)$.

The intersection $L_1$   $L_2$ consists of the subspace of fixed vectors of the operators $H_{12}$ and $H_{21}$ and the null subspace of the operators $\Phi_{12}$ and $\Phi_{21}$.

In the case of Hilbert spaces, theorem 4 of the previous paragraph is a partial consequence of the more complete proposition:

Theorem 8. Let the subspaces $L_1$ and $L_2$, whose intersection is the empty set, form a proper pair and let $E = \overline{L_1 + L_2}$. Furthermore, let $\Phi$ - be the inclination operator of $L_1$ onto $L_2$, and $\psi$ - the inclination operator of $N_1$ and $N_2$, where $N_1 = E \quad L_1$, $N_2 = E \quad L_2$. Then dim $N_1$ = dim $N_2$ on the operators $\Phi$ and $\psi$ are unitarily equivalent.

<u>Proof</u>. First of all, we remark that $L_1$ and $N_2$ form a proper pair of subspaces. In fact, the orthogonal complement of $N_2$, i.e., $L_2$ and $L_1$ do not intersect. On the other hand, if $x \in N_2$ is orthogonal to $L_1$, then x is orthogonal to $L_1 + L_2$, i.e., $x = 0$.

It follows from theorem 7 that there exists a spectral operator S which maps $L_1$ and $N_2$ onto each other. Since the operator S is unitary, it maps $N_1$ and $L_2$ onto each other.

By virtue of this unitary property, if $x \in L_1$ and $y = P_{L_2} x$, then $Sx \in N_2$ and $Sy \in N_1$, where

$$Sy = P_{N_1} Sx, \quad SP_{L_2} x = P_{N_1} Sx \qquad (x \in L_1) .$$

Analogously,

$$SP_{L_1} z = P_{N_2} Sz \qquad (z \in L_2) .$$

Consequently,

$$SP_{L_1}P_{L_2}x = P_{N_2}P_{N_1}Sx \qquad (x \in L_1) ,$$

i.e., the operators $\cos^2 \Phi$ and $\cos^2 \psi$ are unitarily equivalent, and therefore the inclination operators $\Phi$ and $\psi$ are also unitarily equivalent.

This concludes the proof of the theorem.

On the basis of the proven theorem, it is easy to point out the relation between the operators $\Phi$ and $\psi$ when $L_1$ and $L_2$ are arbitrary subspaces.

We leave it to the reader to show, with the help of a theorem of Banach [1]), the following two propositions:

1. If the subspaces $L_1$ and $L_2$ form a proper pair, then $PL_1 = L_2$ in the case and only in the case when the maximum angle $\varphi^{(M)}$ between the subspaces $L_1$ and $L_2$ is less than $^{\pi}/2$, i.e., $\theta(L_1,L_2) < 1$.

2. In order that the direct sum of the non-intersecting subspaces $L_1$ and $L_2$ be a complete space, it is necessary and sufficient that the minimum angle $\varphi^{(m)}$ be greater than zero.

We clarify that the minimum angle $\varphi^{(m)}(0 \leq \varphi\ ^{\pi}/2)$ is determined by the formula

$$\cos \varphi^{(m)} = \sup_{\substack{x \in L_1, y \in L_2 \\ \|x\| = \|y\| = 1}} |(x,y)|$$

---

[1]) If a one-to-one mapping of a complete linear subspace into a complete one is continuous, then the inverse mapping is also continuous.

and, since

$$\sup_{y\in L_2,\|y\|=1} |(x,y)| = \|P_{L_2}x\| = \sqrt{(\cos^2 \Phi_{12}x,x)} \qquad (x\in L_1) ,$$

$$\sup_{x\in L_1,\|x\|=1} |(x,y)| = \|P_{L_1}y\| = \sqrt{(\cos^2 \Phi_{21}y,y)} \qquad (y\in L_2) ,$$

then

$$\varphi^{(m)} = \varphi_{12}^{(m)} = \varphi_{21}^{(m)} .$$

3. The whole spectrum of the self-adjoint operator $\Phi = \Phi_{12}$ lies in the interval $(0,{}^{\pi}/2)$, i.e.,

$$0 \le (\cos \Phi x,x) \le (x,x) \qquad (x\in L_1) .$$

There naturally arises the question of whether any self-adjoint operator $\Phi$ acting in some subspace $L_1\subset E$ and satisfying the conditions derived above can be an inclination operator of $L_1$ to some subspace $L_2\subset E$, of course under the condition that dim $(E \ominus L_1)$ is sufficiently large.

The affirmative answer to this question follows directly from a theorem of M. A. Naymark [5,6], according to which we have that, in order that the self-adjoint operator H, which acts on the subspace $L_1\subset E$ and satisfies the condition

$$0 \le (Hx,x) \le (x,x) \qquad (x\in L_1) ,$$

admits the representation

$$Hx = P_{L_1}P_{L_2}x \qquad (x\in L_1) ,$$

where $L_2$ - is some subspace of E, it is necessary, it is necessary and sufficient that

$$\dim (E \ominus L_1) \ge \dim L_1' ,$$

where $L_1'$ - is the collection of vectors in $L_1$, which are orthogonal to all null and fixed vectors of the operator H.

4. Defective indexes of additive and homogeneous operators.

We denote by $\mathcal{D}(A)$ the domain of definition and by $\mathcal{R}(A)$ the range of the additive and homogeneous operator A, acting in a Banach space E.

The point $\lambda_o$ in the complex plane is said to be of the regular type for the operator A, if there exists a positive number $k_{\lambda_o}$ such that

$$\|(A - \lambda_o I)f\| \geq k_{\lambda_o} \|f\| \qquad (f \in \quad (A)) ,$$

where I denotes the identity operator.

The points of regular type for the operator A form an open set, since if $\lambda_o$ is a regular type point for the operator A, then if $|\lambda - \lambda_o| < k_{\lambda_o}$, then

$$\|(A - \lambda I)f\| \geq \|(A - \lambda_o I)f\| - |\lambda - \lambda_o| \cdot \|f\| \geq k_\lambda \|f\| \qquad (f \in \mathcal{D}(A))$$

where

$$k_\lambda = k_{\lambda_o} - |\lambda - \lambda_o| \quad .$$

<u>Theorem 9</u>. Let G be a region in the complex plane, consisting of regular type points for the operator A.

Then the dimensions of all orthogonal complements $N_\lambda$ in E* of $\mathcal{R}(A - \lambda I) \subset E$ are identical for all $\lambda \in G$.

Proof. We shall show that for every point $\lambda_o \in G$ there exists a neighborhood W such that for all $\lambda \in W$

$$\dim N_\lambda = \dim N_{\lambda_o} ,$$

from which the theorem follows.

Let W be a neighborhood of the point $\lambda_o$ of radius $1/4\, k_{\lambda_o}$ ; then for all $\lambda \in W$

$$\|(A - \lambda I)f\| \geq \|(A - \lambda_o I)f\| - |\lambda - \lambda_o|\, \|f\| > 3/4\, k_{\lambda_o} \|f\| \qquad (f \in \mathfrak{D}(A)) ,$$

and

$$\|(A - \lambda I)f - (A - \lambda_o I)f\| = |\lambda - \lambda_o| \cdot \|f\| < 1/3\, \|(A - \lambda I)f\| ,$$

$$\|(A - \lambda_o I)f - (A - \lambda I)f\| < 1/4\, \|(A - \lambda_o I)f\| .$$

Consequently,

$$\theta[\mathfrak{R}(A - \lambda I),\ \mathfrak{R}(A - \lambda_o I)] \leq 1/3 \qquad (\lambda \in W) .$$

By virtue of theorem 5

$$\dim N_\lambda = \dim N_{\lambda_o} .$$

Thus the theorem is proven.

An analogous reasoning, using theorem 6, leads us to the following assertion:

The dimensions of the quotient space $E/\mathfrak{R}(A - \lambda I)$ are identical for all $\lambda$, where $\lambda$ is an element of the connected set of points of regular type for the operator A.

This assertion and theorem 9 give the reason for establishing the concept of defective indices of two types, for each component G of the set of regular type points, for linear operators in a Banach space:

the defective index $m^*$, which equals dim $N_\lambda$ ($\lambda \in G$), and

the defective index $\hat{m}$, which equals dim $E/\mathfrak{R}$ $(A - \lambda I)$.

In case the space is reflective, or in particular when it's a Hilbert space, the defective indices $m^*$ and $\hat{m}$ will coincide.

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13. ABSTRACT

The authors show that the theory of defect indices of Hermitian operators in a Hilbert space can be extended to the case of linear operators in a Banach space. For this purpose they use the idea of aperture (gap) of two sub-spaces with whose help it is convenient to establish in many cases the equality of the dimensions of the subspaces of the Banach space. The authors also arrive at several geometric conjectures.

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